Int. J. Multiphase Flow Vol. 6, pp. 379–382
Pergamon/Elsevier, 1980. Printed in Great Britain

BRIEF COMMUNICATION

COMMENTS TO "SHORT NOTE ON THE SPACE AVERAGING IN CONTINUUM MECHANICS" BY V. N. NIKOLAEVSKII

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(*Received* 22 *May* 1979; *in revised form* 6 *November* 1979)

In the "Short Note", Dr. Nikolaevskii critisizes one very simple but principal affirmation proposed in the book *Fundamentals in Mechanics of Heterogeneous Media* (Nigmatulin 1978), published only in Russian. However this affirmation is also discussed in the article "Spatial averaging in mechanics of heterogeneous.and dispersed mixtures" (Nigmatulin 1979).

Due to some assertions of Dr. Nikolaevskii, I want to restate certain simple rules which always have to be in view when averaging (see the appendix, figures 1 and 2).

Averaging makes sense, and it is always assumed but not always specified, if average values $\langle\varphi'\rangle_S$ and $\langle\varphi'\rangle_V$ are stable and regular. What does this mean? Firstly it means that there is large enough range of dimensions (see figure 1 where a is microscale (the dimension of particles), L is macroscale)

$$\Delta_{\min} < \mathrm{d}x < \Delta_{\max}, \quad a \ll \Delta_{\min} \ll \Delta_{\max} \ll L, \qquad [1]$$

where the values of averaged parameters $\langle\varphi'\rangle_S$ and $\langle\varphi'\rangle_V$ in all points of the flow in question are independent of the dimensions and form of both the plane surface $\mathrm{d}s$ (a circle, square, etc. $\mathrm{d}s \sim (\mathrm{d}x)^2$), and the volume $\mathrm{d}V$ (a sphere, cube, etc. $\mathrm{d}V \sim (\mathrm{d}x)^3$). The values $\langle\varphi'\rangle_S$ and $\langle\varphi'\rangle_V$ (see appendix) that correspond to the plateau in figure 1 are taken as average values in the point r and one may name them stable. It is an obvious condition and usually it is not mentioned.

Secondly, the space distribution of parameters $\langle\varphi'\rangle_S$ and $\langle\varphi'\rangle_V$ just averaged using the mentioned method, is regular or smooth, i.e.

$$\frac{\partial\langle\varphi'\rangle_S}{\partial x} \sim \frac{\partial\langle\varphi'\rangle_V}{\partial x} \sim \frac{\varphi_0}{L}. \qquad [2]$$

To fulfil both these conditions, it is necessary that

$$\langle\varphi'\rangle_S = \langle\varphi'\rangle_V. \qquad [3]$$

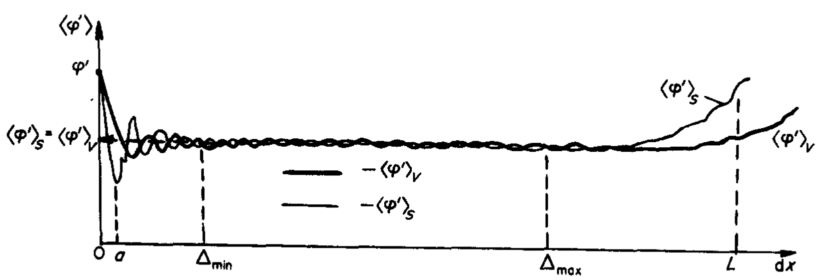


Figure 1.

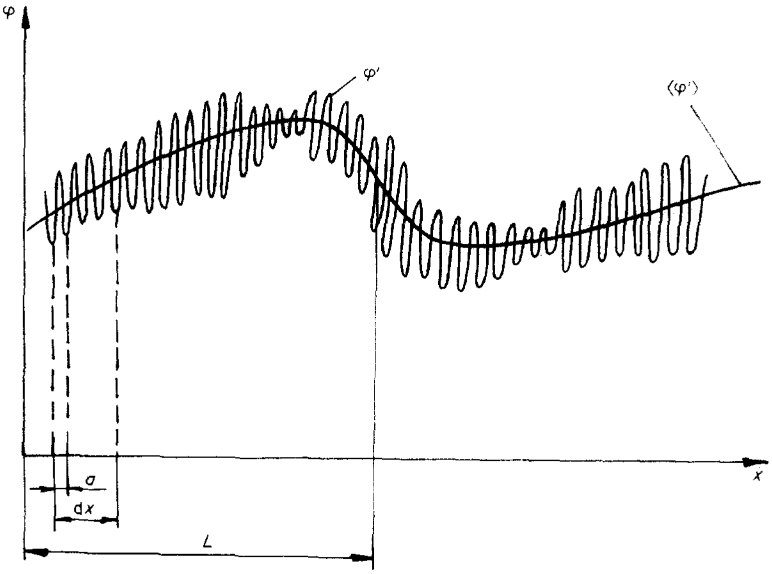


Figure 2.

A simple proof is presented in my book (Nigmatulin 1978) and paper (Nigmatulin 1979) and is reproduced by Nikolaevskii in his "Short note". Nikolaevskii agrees with the fact that

$$\langle\varphi'\rangle_V = \langle\varphi'\rangle_S + \varphi_0 O\left(\frac{\mathrm{d}x}{L}\right). \tag{4a}$$

Moreover, he specifies the assertion showing that

$$\langle\varphi'\rangle_V = \langle\varphi'\rangle_S + \varphi_0 O\left(\left(\frac{\mathrm{d}x}{L}\right)^2\right). \tag{4b}$$

Nevertheless, he assumes that within the framework of the allowances accepted it is possible and, moreover, it is necessary to take into consideration the differences between $\langle\varphi'\rangle_S$ and $\langle\varphi'\rangle_V$. Then naturally to ask author of "Short Note" to give an example of distribution of $\langle\varphi'\rangle_V$ and $\langle\varphi'\rangle_S$ when $\langle\varphi'\rangle_V = \langle\varphi'\rangle_S$. In connection with equation of an internal moment of momentum

$$\frac{\partial M^i}{\partial t} + \frac{\partial M^i v^j}{\partial x^j} = \epsilon^{ikl}\langle\sigma'^{kl}\rangle_S + \frac{\partial \mu^{ij}}{\partial x^j} + c^i, \tag{5}$$

he says: "So we cannot neglect the value of an order of $O(\mathrm{d}x/L)$ because the value of an order of 0(1) is absent. The stress $\epsilon^{ikl}\langle\sigma'^{kl}\rangle_S$, which is included in balance [5] has the order of $O(\mathrm{d}x/L)$, that is *proportional to the dimension of the elementary macro-volume*" (?!). This is a very strange statement. Equation [5] is the averaged equation and has only averaged values which do not depend on dx, i.e. the dimension of elementary macrosurface ds or elementary macro-volume dV (see figure 1). In the contrary case [5] has no meaning.

If we use different $\langle\varphi'\rangle_S$ and $\langle\varphi'\rangle_V$ values, then these will oscillate at distances $\lesssim$dx, or will be unstable and irregular (i.e. do not satisfy the above mentioned conditions [1] and [2]). This is a consequence of insufficiently representative ds and dV.

After reading the article by Nikolaevskii, one can make a conclusion that in my book, antisymmetrical stresses are not permitted. (Dr. Nikolaevskii asks a rhetorical question: "Is the asymmetry of stress tensor imaginary?"

In fact, in the book (Nigmatulin 1978) and paper (Nigmatulin 1979) the following is asserted.

If the aformentioned hypotheses concerning the stability and regularity of the averages with respect to ds and dV are valid, then due to [3], $\langle\sigma'^{kl}\rangle_S = \langle\sigma'^{kl}\rangle_V = \langle\sigma'^{kl}\rangle = \langle\sigma'\rangle^{kl}$, and if $\sigma'^{kl} = \sigma'^{lk}$ then $\langle\sigma'\rangle^{kl} = \langle\sigma'\rangle^{lk}$.

But instead of $ds = ds_1 + ds_2$ or ds_1 (see figure 3a),

$$\langle\sigma_1'\rangle_1{}^{kl} = \frac{1}{ds_1}\int_{ds_1} \sigma_1'^{kl}\, d's, \tag{6}$$

when it is possible to introduce another surface of averaging $ds_{1*} = ds_1 + ds_{21S}$ (see figure 3b), so that the reduced stress tensor

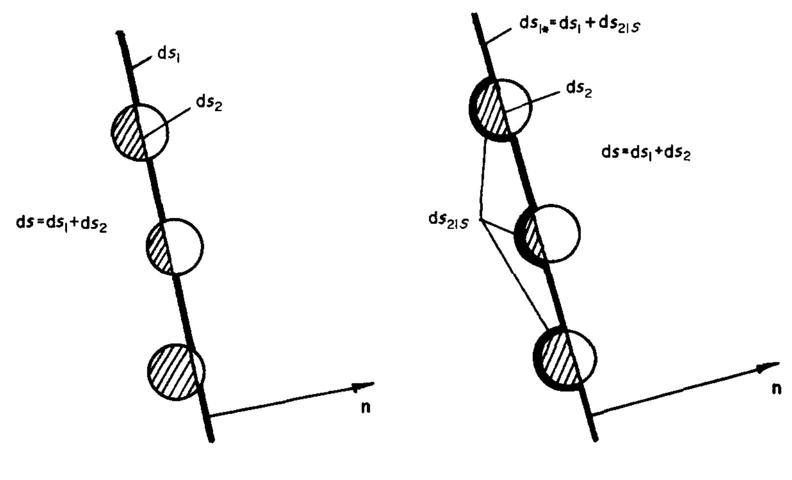
$$\boldsymbol{\sigma}_{1*}^{n} = \frac{1}{ds(\mathbf{n})}\int_{ds_{1*}} \boldsymbol{\sigma}_1'^{k}\mathbf{n}_1'^{k}\, d's = \alpha_1\langle\boldsymbol{\sigma}_1'\rangle_1{}^{n} + \boldsymbol{\sigma}_{21S}^{n}, \tag{7}$$

$$\boldsymbol{\sigma}_{21S}^{n} = \frac{1}{ds(\mathbf{n})}\int_{ds_{21S}} \boldsymbol{\sigma}_1'^{k}\mathbf{n}_1'^{k}\, d's, \tag{8}$$

involving σ_{21S}^{kl} describing a part of interphase force along the portion of interphase boundary ds_{21S} can be non symmetrical (for the oriented rotation of particles) even if $\sigma_1'^{kl} = \sigma_1'^{lk}$ (see figure 4). The tensor σ_{1*}^{kl} is widely used in my book and paper (see [8.6], [11.16], [12.21], [12.42] in the paper). Naturally, correctly written equations using either $\langle\sigma_1'\rangle_1{}^{kl}$ and σ_{1*}^{kl} as a result must be the same. The differences between these two methods are purely methodical.

$$\overbrace{\nabla^k\alpha_1\langle\boldsymbol{\sigma}_1'^{k}\rangle_1 + \underbrace{\nabla^k\boldsymbol{\sigma}_{21S}^{k} + n\mathbf{f}_{21}}_{\mathbf{R}_{21}}}^{\nabla^k\sigma_{1*}^{k}} = \begin{cases} \nabla^k\alpha_1\langle\boldsymbol{\sigma}_1'^{k}\rangle_1 + \mathbf{R}_{21}, & [9a] \\ \nabla^k\boldsymbol{\sigma}_{1*}^{k} + n\mathbf{f}_{21}. & [9b] \end{cases}$$

Here α_1 is the volume concentration of carrier phase, n is the number concentration of particles, $\mathbf{R}_{21}$ is the interphase force, $\mathbf{f}_{21}$ is the force from one dispersed particle. The difference in definition of the tensors $\langle\sigma_1'\rangle_1{}^{kl}$ and σ_{1*}^{kl} is essential for their *calculation*, when a scheme of micromotion is given. If, aside from general words about averaging, one makes an attempt to obtain (derive) rheological relations based on *schematization of micromotion*, one would see these circumstances. It is the procedure that was considered and made in my book. Short results one can find in the paper "Spatial averaging".



(a) (b)

Figure. 3.

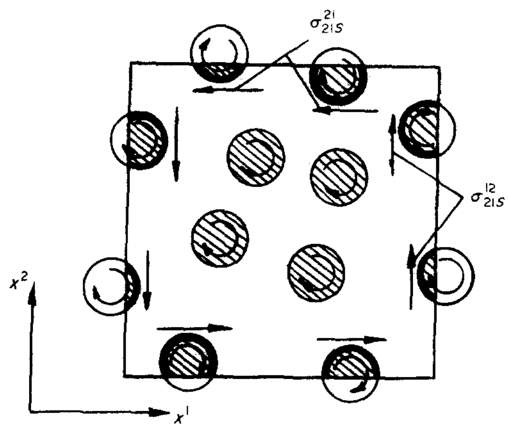


Figure 4.

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APPENDIX

(1) ds is the elementary plane macrosurface ($ds \sim (dx)^2$) with the center defined by the radius-vector

$$\mathbf{r} = \frac{1}{ds}\int_{ds} \mathbf{r}' d's.$$

(2) dV is elementary macrovolume ($dV \sim (dx)^3$) with the center defined by the radius-vector

$$\mathbf{r} = \frac{1}{dV}\int_{dV} \mathbf{r}' d'V.$$

$$\langle\varphi'\rangle_S = \frac{1}{ds}\int_{ds} \varphi' \, d's$$

(3) is the average value at the plane surface ds referred to the point $\mathbf{r}$;

$$\langle\varphi'\rangle_V = \frac{1}{dV}\int_{dV} \varphi' \, d'V$$

(4) is the average value over the volume dV referred to the point $\mathbf{r}$.